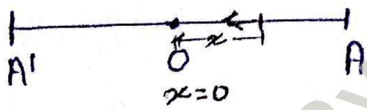
Simple harmonic motion (SHM)

- Periodic motion :- Uniform circular motion.
- Oscillation - Motion of pendulum
- SHM : Linear SHM / Angular SHM - simple pendulum

Definition :- If a particle moves in such a way that accln of the particle is proportional to its displacement from a fixed point and in the direction opposite to it than object will under go an oscillation known as simple Harmonic motion.



$x = 0$
$F = 0$

O - point is stable eqm

$\boxed{\vec{a} \propto -\vec{x}}$ Condition of SHM

In terms of force.

$\boxed{\vec{F} \propto -x}$

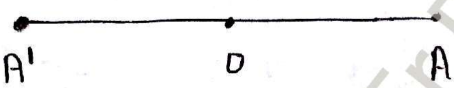
Some important term about S.H.M

1. Mean position (O) - point at which $\vec{F}_{net} = 0$ is called mean position.
2. Amplitude :- Maximum displacement from mean position.
3. Time period :- Time taken to complete one cycle.
4. Frequency :- $f = \frac{1}{T}$

5. Angular frequency :- $\omega = 2\pi f = \frac{2\pi}{T}$

$x = -A$	0	$x = A$
$\lvert a \rvert = max^m$	$a = 0$	$\lvert a \rvert = max^m$
$\lvert \vec{v} \rvert = 0$	$v = max^m$	$\lvert \vec{v} \rvert = 0$
$KE = 0$	$KE = max^m$	$KE = 0$
$PE = max^m$	$PE = 0$	$PE = max^m$

Equation of motion of particle Executing SHM :-



$\vec{a} \propto -\vec{x}$

$a = -\omega^2 x$

$\frac{v\,dv}{dx} = -\omega^2 x$

$\Rightarrow v\,dv = -\omega^2 x\,dx$

integrating both side.

$\Rightarrow \int v\,dv = -\omega^2 \int x\,dx$

$\frac{v^2}{2} = -\frac{\omega^2 x^2}{2} + C$

at $x = A$, $v = 0$

$0 = -\omega^2 \frac{A^2}{2} + C$

$C = \frac{+\omega^2 A^2}{2}$

$\frac{v^2}{2} = \frac{+\omega^2 A^2}{2} - \frac{\omega^2 x^2}{2}$

$v^2 = \omega^2 (A^2 - x^2)$

$\boxed{v = \omega \sqrt{A^2 - x^2}}$

$\frac{dx}{dt} = \omega \sqrt{A^2 - x^2}$

$\frac{dx}{\sqrt{A^2 - x^2}} = \omega \cdot dt$

integrating both side

$\int \frac{dx}{\sqrt{A^2 - x^2}} = \int \omega \cdot dt$

$\sin^{-1}\left(\frac{x}{A}\right) = \omega t + \phi$

$\boxed{x = A\sin(\omega t + \phi)}$

If at $t = 0$ $x = x_0$

$$\boxed{\sin^{-1}\left(\frac{x_0}{A}\right) = \phi}$$

Initial phase of the particle.

time period of this eqn will be

$$T = \frac{2\pi}{\omega}$$

$$\boxed{\omega = \frac{2\pi}{T}}$$ angular frequency of S.H.M

Let. $$\boxed{x = A\sin(\omega t + \phi)}$$

(1) $T = 2\pi/\omega$

(2) $\phi = \sin^{-1}\left(\frac{x_0}{A}\right)$ where x_0 is displacement at $t=0$

(3) Any function $f(t)$ will represent SHM of it is sine or cosine function of time 't'

(4) $v = \frac{dx}{dt} = A\omega\cos(\omega t+\phi)$ — $v_{max} = +A\omega$, $v_{min} = -A\omega$

(5) $a = \frac{d^2x}{dt^2} = -A\omega^2\sin(\omega t+\phi)$ — $a_{max} \cdot A\omega^2$, a_{min} $-A\omega^2$ $|a| = A\omega^2$

(6) $v^2 = \omega^2(A^2 - x^2)$

How to calculate time period of a Linear S.H.M

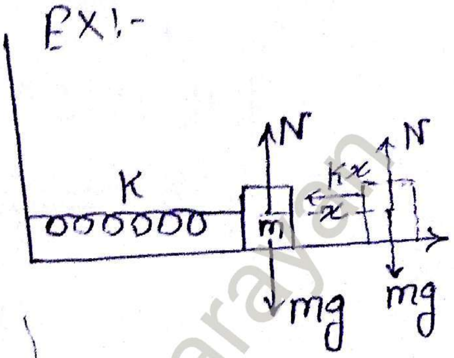
Step 1 → Find mean position of SHM {i.e point at which $F_{net} = 0$}

Step-2 :- displaced the object by a small distance x from its mean position and calculate F_{net} and hance accln $\vec{a}$

Step-3 :- If $\vec{a} \propto -x$ then particle will excute SHM and hence equate it ~~stan~~ with $a = -\omega^2 x$ and find ω and hence 'T'

$$\boxed{T = 2\pi\sqrt{\frac{m}{K}}}$$

Ex :-



$\vec{F}_{net} = -Kx$

$\vec{a} = -\frac{K}{m}\vec{x}$ —(I)

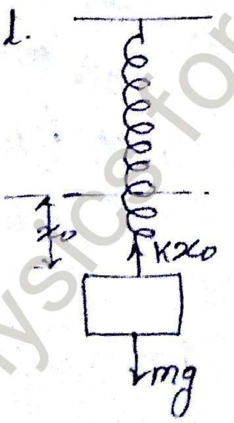
$\because \vec{a} \propto -x$

$a = -\omega^2 x$ —(II)

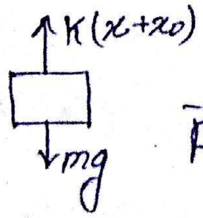
$\omega = \sqrt{K/m}$

$$\boxed{T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{K}}}$$

Note :- 1.



$Kx_0 = mg$

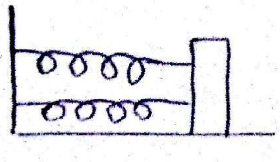


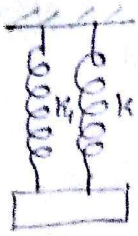
$\vec{F}_{net} = -K\vec{x}$

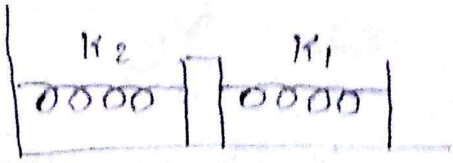
$\vec{a} = \frac{-K}{m}\vec{x}$

$$\boxed{\therefore T = 2\pi\sqrt{\frac{m}{K}}}$$

Note-2 :- Spring connected in parallel.







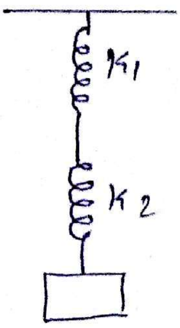
$$\vec{F}_{net} = -(K_1 + K_2)\vec{x}$$

$$\boxed{K_{eq} = K_1 + K_2}$$

$$\boxed{\therefore\ T = 2\pi\sqrt{\frac{m}{K_{eq}}}}$$

$$\boxed{K_{eq} = K_1 + K_2 \cdots + K_n}$$

Note-3:- Series Combination.



$$x_1 + x_2 = x \quad \text{—— (I)}$$

$$F_{net} = K_2\vec{x} \quad \text{—— (II)}$$

$$K_1 x_1 = K_2 x_2 \quad \text{—— (III)}$$

Solving (I) and (III)

$$\frac{K_2}{K_1}x_2 + x_2 = x$$

$$\boxed{\therefore\ x_2 = \frac{K_1 x}{K_1 + K_2}}$$

$$\therefore\ F_{net} = \frac{-K_1K_2}{K_1+K_2}\cdot x$$

$$\boxed{K_{eq} = \frac{K_1K_2}{K_1+K_2}}$$

$$\boxed{\therefore \frac{1}{K_{eq}} = \frac{1}{K_1} + \frac{1}{K_2} \cdots + \frac{1}{K_n}}$$

$$\boxed{\therefore\ T = 2\pi\sqrt{\frac{m}{K_{eq}}}}$$

Note-4 :-

$$K = \frac{YA}{L}$$

Y = young's modulus of Elasticity

A = Area of cross section.

L = Length of object.

$$Y = \frac{F/A}{\frac{\Delta L}{L}} \quad \therefore \quad \frac{Y\Delta L}{L} = F/A$$

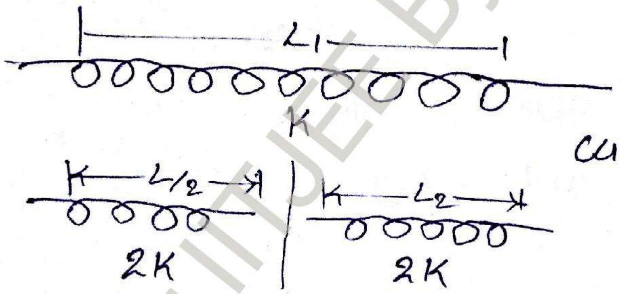
$$F = \frac{Y \cdot A}{L}(\Delta L)$$

$$\boxed{T = 2\pi\sqrt{\frac{m}{K}}}$$

Note-5 :-

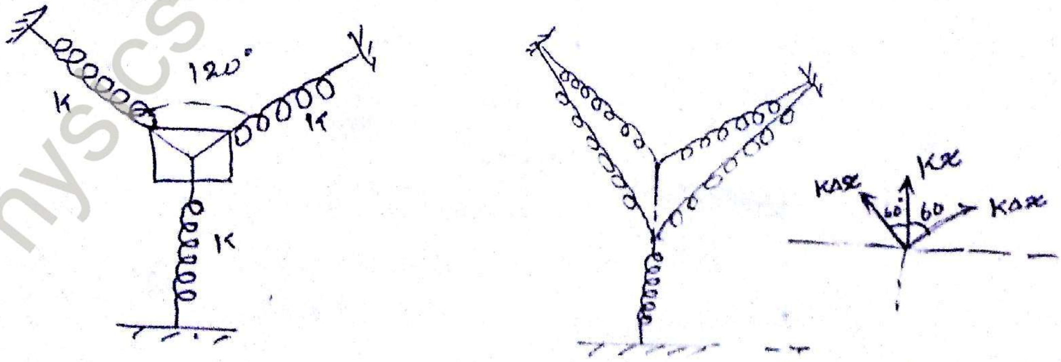
$$K = \frac{YA}{L}$$

$$K \propto \frac{1}{L}$$



cut in to two parts.

Note-6 :-



$$\cos 60 = \frac{Ox}{x}$$

$$Ox = x\cos 60$$

$$= x/2$$

$$\therefore F = -\frac{3}{2}Kx$$

$$\therefore K_{eq} = 3K/2$$

$$F_R = Kx + 2x\cos 60$$

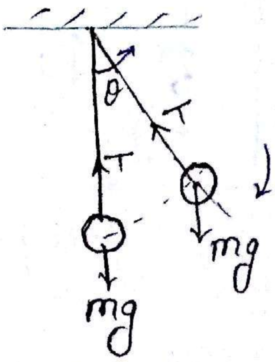
$$= Kx + 2Kx \cdot \frac{1}{2}$$

$$= \frac{3}{2}Kx$$

Angular SHM

Linear SHM	Angular SHM
1. 	1. 
2. $\vec{a} \propto -\vec{x}$ or $\vec{F} \propto -\vec{x}$	2. $\vec{\alpha} \propto -\theta$ $\vec{\tau} \propto -\hat{\theta}$
3. $x = A\sin(\omega t + \phi)$	3. $\theta = \theta_0 \sin(\omega t + \phi)$
4. Time period is calculated by force displacement method.	4. time period is calculated by Torque displacement method.
5. Different spring mass system.	5. Simple pendulum and physical pendulum.

Simple pendulum



$$\tau = mgL\sin\theta$$

$$I\alpha = mgL\sin\theta$$

$$\cancel{m}L^{\cancel{2}}\alpha = \cancel{m}g\cancel{L}\sin\theta$$

$$\alpha = g/L \sin\theta$$

if θ is very small that

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$

$$\therefore \alpha = g/L \cdot \theta$$

$$\alpha = -g/L\,\theta$$

$\because \alpha \propto -\theta$ then it is an angular SHM

$\therefore \omega^2 = g/L$

$$\boxed{\therefore \quad T = 2\pi\sqrt{\frac{L}{g}}}$$

Time period of simple pendulum.

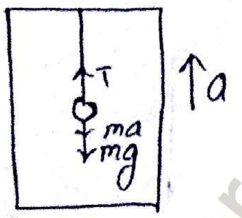
<u>Note 1:-</u> $T = 2\pi\sqrt{L/g}$ should be used for simple pendulum only (i.e. not for other pendulum)

<u>Note-2:-</u> If $T = 2$ sec, then it is known as second's pendulum.

<u>Note-3:-</u> Time period of simple pendulum

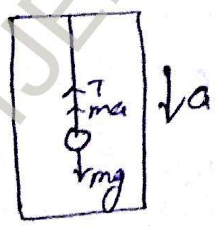
$$\boxed{T = 2\pi\sqrt{\frac{L}{g_{eff}}}}$$

<u>Note-4:-</u> Simple pendulum in accelerated frame of Reference.



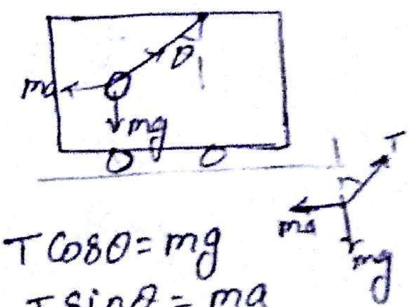
$$\boxed{g_{eff} = g + a}$$

$$T = 2\pi\sqrt{\frac{L}{g+a}}$$



$g_{eff} = g - a$

$$\boxed{T = 2\pi\sqrt{\frac{L}{g-a}}}$$

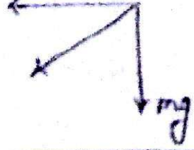


$T\cos\theta = mg$

$T\sin\theta = ma$

$\tan\theta = a/g$

$\theta_0 = \tan^{-1}(a/g)$

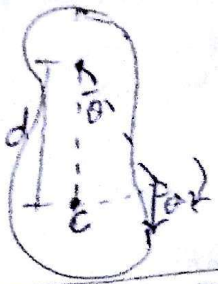


$m\sqrt{g^2+a^2}$

$$\boxed{T = 2\pi\sqrt{\frac{L}{\sqrt{g^2+a^2}}}}$$

Note-6:-

physical pendulum
or
Compound pendulum



$$\boxed{T = 2\pi\sqrt{\frac{I}{mgd}}}$$

I = moment of inertia about hinged point.

m = mass of object

d = vertical separation between point of suspension and CG of the object.

$$\tau = mg\sin\theta \cdot d$$

$$I\alpha = mgd\sin\theta$$

$$\alpha = \frac{mgd\sin\theta}{I}$$

If $\theta \approx 0$

$$\alpha = -\frac{mgd \cdot \theta}{I}$$

Hear $\omega^2 = \frac{mgd}{I}$

$$\boxed{\therefore T = 2\pi\sqrt{\frac{I}{mgd}}}$$

Energy of particle Executing SHM

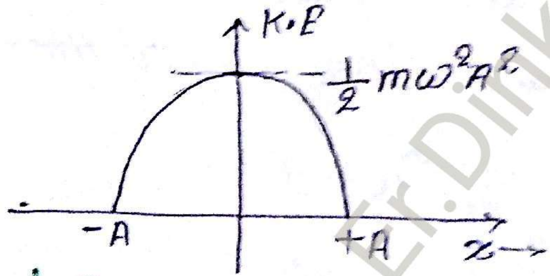
→ Kinetic Energy:-

$$KE = \frac{1}{2}mv^2 \quad \text{---} (I)$$

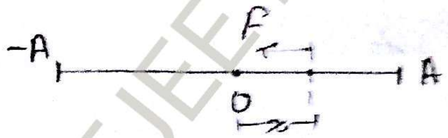
$$v^2 = \omega^2(A^2 - x^2) \quad \text{---} (II)$$

$\therefore$ from (I) and (II)

$$\boxed{KE = \frac{1}{2}m\omega^2(A^2 - x^2)}$$



→ potential Energy:-



From difinition of potential Energy

$$\Delta U = W \qquad\qquad a = -\omega^2 x$$

$$= \int dW$$

$$= \int F \cdot dx$$

$$= -\int ma\,dx$$

$$= \int_0^x m\omega^2 x\,dx$$

$$= m\omega^2\left[\frac{x^2}{2}\right]_0^x$$

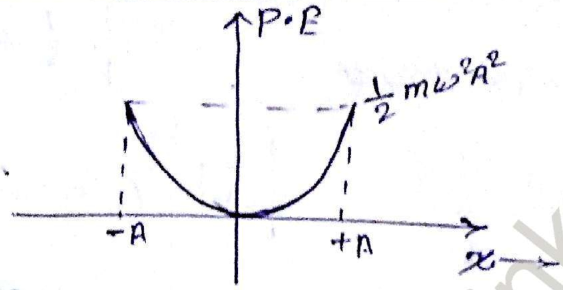
$$\boxed{\Delta U = \frac{m\omega^2 x^2}{2}}$$

If potential Energy at mean position is zero

$$\boxed{U = \frac{1}{2} m\omega^2 x^2}$$

If potential Energy at mean position is U_0

$$\boxed{U = U_0 + \frac{1}{2} m\omega^2 x^2}$$



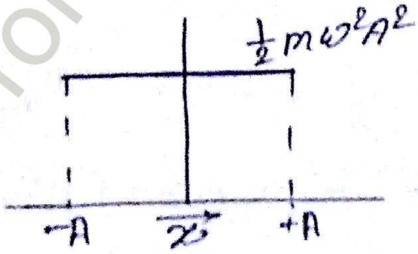
→ Total Energy :-

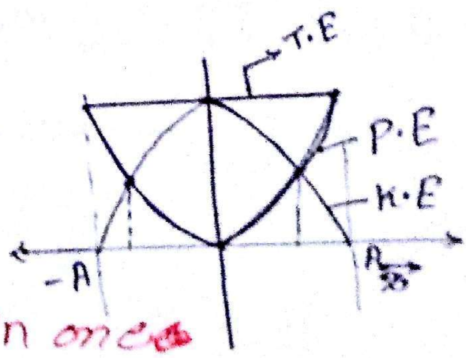
$$E = K.E + P.E$$

$$= \frac{1}{2} m\omega^2 (A^2 - x^2) + \frac{1}{2} m\omega^2 x^2$$

$$= \frac{1}{2} m\omega^2 A^2 - \cancel{\frac{1}{2} m\omega^2 x^2} + \cancel{\frac{1}{2} m\omega^2 x^2}$$

$$\boxed{E = \frac{1}{2} m\omega^2 A^2}$$





Average value of K.E and P.E in one Complete cycle :-

Avg. Kinetic Energy

$$K.E = \frac{1}{2} m\omega^2 (A^2 - x^2) \quad \text{—(I)}$$

$$x = A\sin(\omega t + \phi) \quad \text{—(II)}$$

Avg. potential Energy

$$U = \frac{1}{2} m\omega^2 x^2 \quad \text{—(I)}$$

$$x = A\sin(\omega t + \phi) \quad \text{—(II)}$$

$$(KE)_t = \frac{1}{2} m\omega^2 \{A^2 \cos^2(\omega t+\phi)\}$$

$$= \frac{1}{2} m\omega^2 A^2 \cos^2(\omega t+\phi)$$

$$\therefore (KE)_{Avg} = \frac{\int_0^T KE_{(t)} dt}{T}$$

$$= \frac{\omega}{2\pi} \int_0^T \frac{1}{2} m\omega^2 A^2 \cos^2(\omega t+\phi) dt$$

$$= \frac{\omega}{2\pi} \frac{1}{2} m\omega^2 A^2 \int_0^T \cos^2(\omega t+\phi) dt$$

$$\boxed{(KE_{Avg}) = \frac{1}{4} m\omega^2 A^2}$$

$$(U)_t = \frac{1}{2} m\omega^2 A^2 \sin^2(\omega t+\phi)$$

$$(U_{Avg}) = \frac{\int_0^T (U)_t \cdot dt}{T}$$

$$= \frac{1}{T} \int_0^T \frac{1}{2} m\omega^2 A^2 \sin^2(\omega t+\phi) dt$$

$$= \frac{1}{T} \frac{1}{2} m\omega^2 A^2 \int_0^T \sin^2(\omega t+\phi) dt$$

$$\boxed{\langle PE \rangle = \frac{1}{4} m\omega^2 A^2}$$

Q. A mass 0.2 kg is executing SHM along x-axis with a frequency of $\frac{25}{\pi}$ Hz. If the K.E of the particle at $x = 0.04$ m is 0.5 J and PE is 0.4 J then calculate the amplitude of S.H.M.

Soln:- We know that

$$KE = \frac{1}{2} m\omega^2 (A^2 - x^2)$$

$$PE = \frac{1}{2} m\omega^2 x^2 + U_0$$

$$(KE)_{x=0.5} = 0.5 J = \frac{1}{2} 0.2 \times \left(2\pi \cdot \frac{25}{\pi}\right)^2 \left[A^2 - (0.04)^2\right]$$

$$= 0.1 \times 2500 \{A^2 - 16\times10^{-4}\}$$

$$= 250 \{A^2 - 16\times10^{-4}\}$$

$$\Rightarrow A^2 - 16\times10^{-4} = \frac{0.5}{250} = 0.2\times10^{-2}$$

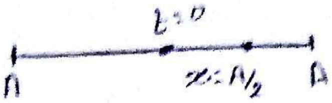
$$\Rightarrow A^2 - 16\times10^{-4} = 20\times10^{-4}$$

$$A^2 = 36\times10^{-4}$$

$$A = 6\times10^{-2} = 0.06 m \text{ Ans}$$

Q. A particle executing SHM with time period of 24 s. Find the time taken by the particle in order to move from mean position to a distance half of its amplitude.

Soln



$$x = A\sin(\omega t)$$

$$A/2 = A\sin(\omega t)$$

$$\sin\omega t = 1/2$$

$$\omega t = \pi/6$$

$$t = \frac{\pi/6}{\omega} = \pi/6\omega$$

$$T = 24 \text{ sec.}$$

$$\Rightarrow \frac{2\pi}{\omega} = 24 \text{ sec.}$$

$$\Rightarrow \frac{\pi}{\omega} = 12 \text{ sec.}$$

$$\therefore t = \frac{\pi}{6\omega} = \frac{1}{6}\times \cancel{12}^{2} = \underline{2 \text{ sec.}}$$ Ans

Q. If speed (v) and displacement (x) of a particle executing SHM are related with the equation given by $4v^2 = 25 - x^2$ then find the time period and Amplitude of S.H.M.

Soln

$$4v^2 = 25 - x^2$$

$$v^2 = \frac{1}{4}(25 - x^2) \quad \text{——} \ (I)$$

$$v^2 = \omega^2(A^2 - x^2) \quad \text{——} \ (II)$$

equating (I) with (II)

$\Rightarrow \omega^2 = \frac{1}{4}$

$\Rightarrow \omega = \frac{1}{2}$

$\therefore T = 2\pi/\omega$

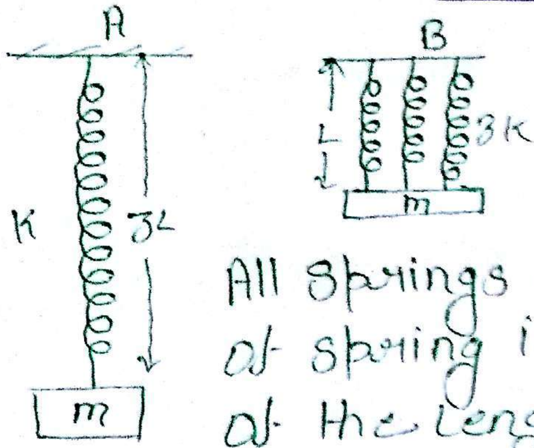
$= 2\pi/\frac{1}{2}$

$= 4\pi$ sec Ans

$A^2 = 25$

$A = 05$ m Ans

Q.



All springs are identical but lengths of spring in system B is $\frac{1}{3}$rd that of the length of system A. Find the ratio of $\frac{T_a}{T_b}$

Soln

$K \propto \frac{1}{L}$

Time period

$T_A = 2\pi\sqrt{\frac{m}{K}}$ —— (I)

$T_B = 2\pi\sqrt{\frac{m}{K_{eq}}}$

$T_B = 2\pi\sqrt{\frac{m}{9K}}$ —— (II)

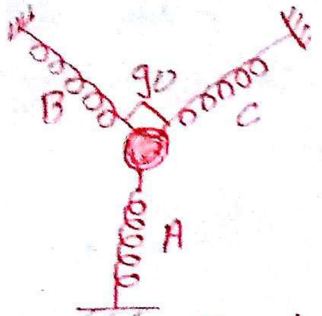
$[\because K_{eq} = 3K + 3K + 3K = 9K]$

$\therefore \frac{T_A}{T_B} = \frac{2\pi\sqrt{\frac{m}{K}}}{2\pi\sqrt{\frac{m}{9K}}}$

$= \sqrt{9}$

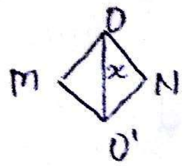
$\boxed{\therefore \frac{T_A}{T_B} = 3}$ Ans

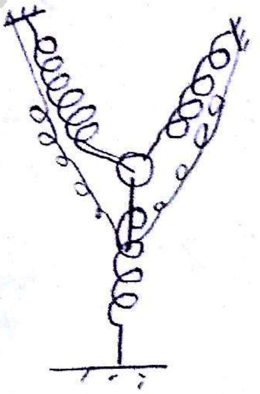
Q. A particle of mass 'm' is attached with three identical spring of spring constant 'K' as shown in fig.



If the mass is slightly pushed against spring 'A' and released then find the time period of resulting SHM.

Sol^n





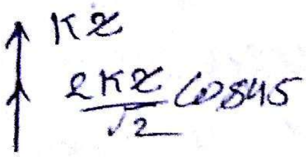
In $\Delta OO'M$

$OM = OO' \cos 45 = x \cdot 1/\sqrt{2}$

$\boxed{OM = 1/\sqrt{2}}$

$F = Kx + \frac{2Kx}{\sqrt{2}} \cos 45^\circ$

$= Kx + 2 \cdot \frac{Kx}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}$



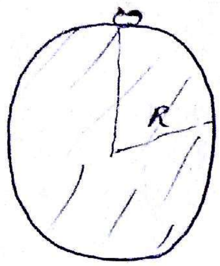
$F = 2Kx$

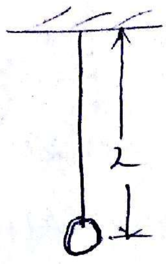
$K_{eq} = 2K \qquad T = 2\pi\sqrt{\frac{M}{K_{eq}}}$

$\boxed{T = 2\pi\sqrt{\frac{m}{2K}}}$ Ans

Q. A disc of mass 'M' and radius 'R' is set to oscillate in a vertical plane hinged at its rim. What should be the length of simple pendulum if time period of simple pendulum and disc are equal.

Soln





$$T = 2\pi\sqrt{\frac{I}{mgd}}$$

$$= 2\pi\sqrt{\frac{MR^2/2 + MR^2}{Mg \cdot R}}$$

$$= 2\pi\sqrt{\frac{\frac{3}{2}\cancel{M}R^{\cancel{2}}}{\cancel{M}g\cancel{R}}}$$

$$= 2\pi\sqrt{\frac{3R}{2g}} \quad \text{—(I)}$$

$$T = 2\pi\sqrt{L/g} \quad \text{—(II)}$$

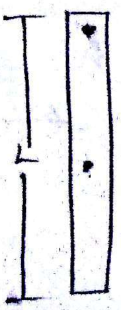
from (I) and (II)

$$\cancel{2\pi}\sqrt{L/g} = \cancel{2\pi}\sqrt{\frac{3R}{2g}}$$

$$\boxed{L = \frac{3R}{2}}$$

Q. If a meter stick is oscillating with small amplitude in a vertical plane about one of its end point is having frequency ν then find its frequency if $\frac{1}{3}$rd length from bottom is cut off.

Soln



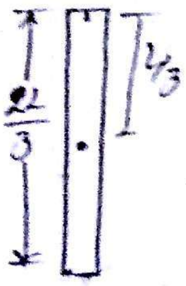
$$T = 2\pi\sqrt{\frac{I}{mg \cdot d}}$$

$$= 2\pi\sqrt{\frac{\cancel{M}L^{\cancel{2}}/3}{\cancel{M}g \cdot \cancel{L}/2}}$$

$$\Rightarrow T = 2\pi\sqrt{\frac{2L}{3g}}$$

$$\therefore \quad f = \frac{1}{2\pi}\sqrt{\frac{3g}{2L}}$$

$$\boxed{\nu_0 = \frac{1}{2\pi}\sqrt{\frac{3g}{2L}}}$$



$$m' = \frac{2m}{3}$$

$$T' = 2\pi\sqrt{\frac{I}{m'g \cdot d}}$$

$$= 2\pi\sqrt{\frac{\frac{m'\left(\frac{2L}{3}\right)^2}{3}}{m'g \cdot L/3}}$$

$$= 2\pi\sqrt{\frac{4L^{\cancel{2}}}{9g \times \cancel{L}}}$$

$$= 2\pi\sqrt{\frac{4L}{9g}}$$

$$= \frac{2}{3}2\pi\sqrt{L/g}$$

$$f = 3/2 \, \frac{1}{2\pi}\sqrt{g/L}$$

$$= 3/2 \, \nu_0 \sqrt{2/3}$$

$$f = \nu_0 \, 3/2 \sqrt{2/3}$$

$$= \nu_0\sqrt{\frac{\cancel{2}}{\cancel{3}} \times \frac{\cancel{9}^{\,3}}{\cancel{4}_{\,2}}} = \nu_0\sqrt{3/2}$$

$$= \nu_0\sqrt{1.5} = \underline{1.22\,\nu_0 \quad \text{Ans}}$$